EVENT DRIVEN CONTROL OF SWITCHED-INTEGRATOR-SYSTEMS

LE CONTROLE EVENEMENTIEL DES SYSTEMES D'INTEGRATEURS

Thomas MOOR

University of the Federal Armed Forces
D - 22039 Hamburg, Germany
Tel: ++49-40-6541 2121, E-mail: thomas.moor@unibw-hamburg.de

Abstract: The paper considers hybrid systems consisting of an n-dimensional switched integrator, generating discrete events by comparing the state variables with fixed values. A condensed model is established, suitable for locating cyclic trajectories. A control law is set up, achieving a given cyclic trajectory to become globally attractive w.r.t. the discrete events.

Résumé: Les systèmes hybrides traités dans cette communication sont constitués d'intégrateurs à commutation de dimension n. Des événement discrets sont générés en comparant l'état continu avec des valeurs fixes. Un modèle condensé permet de déterminer les trajectoires cycliques qui, par l'intermédiaire de la loi de commande, peuvent atteindre des cycles déterminés vis-à-vis des événements discrets.

1. INTRODUCTION

Switched-integrator-systems exhibit a continuous state evolving along continuous flow of time. The involved continuous dynamics is represented by an n-dimensional integrator. Furthermore switched-integrator-systems provide a discrete interface to the environment, acting on input-events by switching the integrand and generating output-events by comparing the continuous state with fixed values. When closing the loop by a feedback law in form of a finite automaton in general both discrete and continuous dynamics are involved. Therefore the closed loop is a typical hybrid system. Even if the continuous part as well as the discrete part are set up in a simple manner, the closed loop may give rise to complex dynamics. As an example take the widely discussed switched-server-arrival systems, which are contained in the proposed system class. See i.e. [CHA 93] and [YU 96].

In the past decade a number of different frameworks have been developed, in order to analyze hybrid systems. See [BRA 96] for a comparatively universal concept. [ENG 97] highlights real applications of hybrid character. On principle switched-integrator-systems are covered by hybrid automata, as introduced in [ALU 93]. However, in the present contribution a more explicit notation is prefered.

We first focus on the open system and state as control problem the construction of a feedback law, achieving all trajectories to reach a given cycle w.r.t. the discrete events. In order to solve this problem, a finite automaton, not necessarily deterministic, is proposed as a condensed model. The aim is, to cover a certain amount of the external behavior of the underlying switched-integrator-system, just enough to be able to solve the control problem. The condensed model is extended for the ability of handling certain situations, where the number of controls required to reach the cycle is unbounded. An implementation of the proposed algorithms based on 'Mathematica' has been used to analyze the example given in the last section.

Section 2 gives a suitable definition of switched-integrator-systems and establishes the continuous and discrete transfer functions. From those in section 3 the condensed model is developed. Section 4 states criteria for the existence of cyclic trajectories. In Section 5 an algorithm is provided, setting up the claimed feedback. In Section 6 an extension of the condensed model is proposed. Section 7 gives an example.

2. SWITCHED-INTEGRATOR-SYSTEMS

Let a continuous process be modeled by an n-dimensional integrator with state variable $x \in \mathbb{R}^n$. As input consider a piecewise constant function u, valued within a finite set of inputs $U \subset \mathbb{R}^n,\ |U| \in \mathbb{N}$. As output of the system take the series of events $\mathbf{y} = (y_k)_k$, generated by comparing the components $x^{(i)}$ of the state with finite sets of fixed values $S_i \subset \mathbb{R},\ |S_i| \in \mathbb{N}$. After a suitable substitution of the state variable x (which may increase the states dimension n) the situation can be reduced to comparing the components $x^{(i)}$ with zero. Therefore $y_k = (\operatorname{sign} x^{(i)})_{1 \le i \le n}$ is taken as output event, occuring at time t_k if for an i the $\operatorname{sign} x^{(i)}$ becomes zero at time t_k. For simplicity of notation we only treat inputs producing an infinite sequence of output events. Further assume that never more than one component of x becomes zero simultaneously. Since we are looking for a control law to generate the input from the output events, we restrict the input to change its value only when an output event occurs. We now view the input u as a series of input events $\mathbf{u} = (u_k)_k$, where the event $u_k \in U$ occurs at time t_k. The following definition describes the scenario:

Definition 1. Let $U,\ |U| \in \mathbb{N}$, be a finite subset of $\mathbb{R}^n$. Then the triple $\Sigma_{sis} = (U,\ \mathbb{R}^n,\ \{-1,0,1\}^n)$ is said to be a *switched-integrator-system*. A triple $(\mathbf{u},\ \mathbf{x},\ \mathbf{y})$ consisting of

$$\mathbf{u} = (u_i)_{i\in\mathbb{N}_0}, \quad u_i \in U\ , \tag{1}$$
$$\mathbf{x} = (x_i)_{i\in\mathbb{N}_0}, \quad x_i \in \mathbb{R}^n\ , \tag{2}$$
$$\mathbf{y} = (y_i)_{i\in\mathbb{N}_0}, \quad y_i \in \{-1,0,1\}^n\ , \tag{3}$$

is said to be a *solution* of Σ_{sis} if there exists

$$\mathbf{t} = (t_i)_{i\in\mathbb{N}_0}, \quad t_i \in \mathbb{R}_0^+\ , \tag{4}$$
$$x\colon \mathbb{R}_0^+ \to \mathbb{R}^n\ , \tag{5}$$
$$y\colon \mathbb{R}_0^+ \to \{-1,0,1\}^n\ , \tag{6}$$

such that the following equations hold for all $k \in \mathbb{N}_0$:

$$t_0 = 0, \quad x(0) = x_0\ , \tag{7}$$
$$\dot{x}(t) = u_k \in U \quad \forall\, t \in [t_k, t_{k+1}[\ , \tag{8}$$
$$y(t) = (\operatorname{sign} x^{(i)}(t))_{1\le i\le n}\ , \tag{9}$$
$$t_{k+1} = \sup\{t|\ t > t_k :\ y \text{ is const. on }]t_k, t[\}, \tag{10}$$
$$y_k = y(t_k), \quad x_k = x(t_k)\ , \tag{11}$$
$$x^{(j)}(t) = 0 \quad \Longrightarrow \quad x^{(i)}(t) \neq 0\ \ \forall\, i \neq j\ . \tag{12}$$

□

If a pair $(x_0,\ \mathbf{u})$ consisting of an initial condition x_0 to the continuous state and a sequence of inputs $\mathbf{u}$ yields a solution, this solution is unique. Further the system is time-invariant. We prove both statements by setting up the continuous transfer function $\Phi_{cont}(\cdot,\ u_k)$, which maps any given state x_k to its successor x_{k+1} when input u_k is applied. Pick any solution of Σ_{sis} in the notation of the above definition and fix an arbitrary $k \in \mathbb{N}_0$. By integrating equation (8) we get

$$x(t) = x_k + (t - t_k)u_k \quad \forall\, t \in [t_k, t_{k+1}]\ . \tag{13}$$

For $j,\ 1 \le j \le n$, such that

$$y_k^{(j)} \neq 0 \quad \text{and} \quad y_{k+1}^{(j)} = 0\ , \tag{14}$$

x_{k+1} is the projection of x_k in direction u_k on the hyper-plane $H_j := \{\xi|\ \xi \in \mathbb{R}^n,\ \xi^{(j)} = 0\}$. Let $P(u_k,\ j) \in \mathbb{R}^{n\times n}$ denote the according matrix-representation. We now need to express a suitable j in terms of x_k and u_k. From the equations (10), (12) and (13) we observe that a unique j with (14) exists. It is determined by the following conditions:

(i) $0 \neq \operatorname{sign} x_k^{(j)} = -\operatorname{sign} u_k^{(j)}$.

(ii) For all $i,\ 1 \le i \le n,\ i \neq j$ with $0 \neq \operatorname{sign} x_k^{(i)} = -\operatorname{sign} u_k^{(i)}$ it holds

$$-x_k^{(j)} \frac{1}{u_k^{(j)}} < -x_k^{(i)} \frac{1}{u_k^{(i)}}\ . \tag{15}$$

Changing the point of view we fix an arbitrary $j,\ 1 \le j \le n$. The conditions (i) and (ii) then describe the set of x_k, for which equation (14) holds. Note that these sets are disjunct since j is unique for all x_k. Note also that (i) and (ii) can be written as strict homogeneous linear inequalities. Let $M(y_k,\ u_k,\ j) \in \mathbb{R}^{m\times n}$ denote the according matrix-representation, yielding

$$M(y_k,\ u_k,\ j)x_k > 0 \tag{16}$$
$$\Longrightarrow \quad x_{k+1} = P(u_k,\ j)x_k\ . \tag{17}$$

Define $\Phi_{cont}(x,\ u)$ for all $x \in \mathbb{R}^n,\ u \in U$: If $M(\operatorname{sign} x,\ u,\ j)x > 0$ holds for exactly one $j,\ 1 \le j \le n$, let

$$\Phi_{cont}(x,\ u) := P(u,\ j)x \in \mathbb{R}^n\ . \tag{18}$$

Otherwise let

$$\Phi_{cont}(x,\ u) := 0 \in \mathbb{R}^n\ . \tag{19}$$

Further for all $k \in \mathbb{N}_0,\ (u_i)_{i<k},\ u_i \in U\ \ \forall\, i$, let

$$\Phi^k_{cont}(\cdot,\ (u_i)_{i<k}) := \Phi_{cont}(\cdot,\ u_{k-1}) \circ \cdots \circ \Phi_{cont}(\cdot,\ u_0)\ , \tag{20}$$

$$\Phi^k_{disc}(\cdot,\ (u_i)_{i<k}) := \operatorname{sign} \Phi^k_{cont}(\cdot,\ (u_i)_{i<k}) \in \{-1,0,1\}^n\ . \tag{21}$$

The result of the preceding construction is stated as

Lemma 1. For every solution $(\mathbf{u},\ \mathbf{x},\ \mathbf{y})$ of Σ_{sis} it holds for all $k \in \mathbb{N}_0$:

$$x_k = \Phi^k_{cont}(x_0,\ (u_i)_{i<k})\ , \tag{22}$$
$$y_k = \Phi^k_{disc}(x_0,\ (u_i)_{i<k})\ . \tag{23}$$

For a given pair $(x_0,\ \mathbf{u})$, $x_0 \in \mathbb{R}^n$, $\mathbf{u} = (u_k)_{k\in\mathbb{N}_0}$, $u_k \in U$, define $\mathbf{x} = (x_k)_{k\in\mathbb{N}_0}$ and $\mathbf{y} = (y_k)_{k\in\mathbb{N}_0}$ by the above equations (22) resp. (23). If $x_k \neq 0$ holds for all $k \in \mathbb{N}_0$, then $(\mathbf{u},\ \mathbf{x},\ \mathbf{y})$ is a solution of Σ_{sis}.

□

Note that the restriction allowing only one component of the state to become zero (equation (12)) can be dropped. This will result in a more complicated set of conditions replacing (i) and (ii), still being representable by (not necessarily strict) homogeneous linear inequalities, and therefore still yielding computable transfer functions.

3. CONDENSED MODELS

Let $r \in \mathbb{N}$ be fixed. We develop the discrete *condensed model of order* r by visiting the last r input-events and the last $r+1$ output-events at time t_k for all $k \in \mathbb{N}_0$. As state space choose

$$\mathcal{Z}_r := \{((u_i)_{i<p},\ (y_i)_{i\le p})|\ p \le r\}\ , \qquad (24)$$

where u_i and y_i are in U and $\{-1,0,1\}^n$ respectively. For shortness of notation define for all

$$z = ((u_i)_{i<p},\ (y_i)_{i\le p}) \in \mathcal{Z}_r \qquad (25)$$

the projections

$$\mathfrak{u}_i(z) := u_i, \quad \mathfrak{u}(z) := (u_i)_{i<p}\ , \qquad (26)$$

$$\mathfrak{y}_i(z) := y_i, \quad \mathfrak{y}(z) := (y_i)_{i\le p}\ . \qquad (27)$$

Further define for any pair $(\mathbf{u},\ \mathbf{y})$ of input/output trajectories

$$\mathbf{u} = (u_k)_{k\in\mathbb{N}_0}, \quad u_k \in U\ , \qquad (28)$$

$$\mathbf{y} = (y_k)_{k\in\mathbb{N}_0}, \quad y_k \in \{-1,0,1\}^n\ , \qquad (29)$$

the trajectory $\mathfrak{z}(\mathbf{u},\ \mathbf{y}) = (z_k)_{k\in\mathbb{N}_0},\ z_k \in \mathcal{Z}_r$, by

$$z_k := \begin{cases} ((u_{i+k-r})_{i<r},\ (y_{i+k-r})_{i\le r}) & \text{if } k > r\ , \\ ((u_i)_{i<k},\ (y_i)_{i\le k}) & \text{if } k \le r\ . \end{cases} \qquad (30)$$

We associate $z \in \mathcal{Z}_r$ with the knowledge about the continuous state of the original system Σ_{sis}, for the case that the last r resp. $r+1$ input- and output-events match those collected in z. Let $z \in \mathcal{Z}_r$ be given in the notation of equation (25) and define

$$\mathcal{X}_0(z) := \{x|\ y_{p-j} = \Phi^{p-j}_{disc}(x, (u_i)_{i<p-j})\ \ \forall j \le p\} \setminus \{0\}\ . \qquad (31)$$

Analogous we define $\mathcal{X}_s(z)$ to be the image of $\mathcal{X}_0(z)$ under $\Phi^s_{cont}(\cdot, (u_i)_{i<s})$ for all $s,\ 0 \le s \le p$. Note that $\Phi^{s_2}_{cont}(\cdot, (u_{i+s_1})_{i<s_2})$ restricted on the domain $\mathcal{X}_{s_1}(z)$ is a concatenation of projections (and therefore linear) as long as $s_1 + s_2 \le p$. Note also that the sets $\mathcal{X}_s(z)$ can be computed in a straightforward manner by intersections and linear transformations of cones, namely those represented by the matrices $P(\cdot)$ and $M(\cdot)$ defined in the previous section. Therefor the $\mathcal{X}_s(z)$ are cones also. Taking into account that Φ_{cont} does not depend on k or even t_k (time-invariance), the above definitions imply for all solutions $(\mathbf{u},\ \mathbf{x},\ \mathbf{y})$ of Σ_{sis}, for all $k \in \mathbb{N}_0$ and for all $s_1,\ s_2,\ s_1 + s_2 = \min(r,\ k)$:

$$x_{k-s_2} \in \mathcal{X}_{s_1}(\mathfrak{z}_k(\mathbf{u},\ \mathbf{y}))\ . \qquad (32)$$

When in condensed state $z \in \mathcal{Z}_r$ (again in the notation of (25)) and applying the input $u \in U$, we ask for the set of possible successors $z^+ \in \mathcal{Z}_r$. Therefore define $S(z,\ u)$ to denote the set of all those $z^+ \in \mathcal{Z}_r$, which satisfy the following conditions:

(i) $\mathcal{X}_1(z) \cap \mathcal{X}_0(z^+) \neq \emptyset$.

(ii) It exists a $y \in \{-1,0,1\}^n$ such that:
If $p = r$

$$\mathfrak{u}(z^+) = (u_1,\ \dots,\ u_{r-1},\ u)\ , \qquad (33)$$

$$\mathfrak{y}(z^+) = (y_1,\ \dots,\ y_{r-1},\ y_r,\ y)\ . \qquad (34)$$

If $p < r$

$$\mathfrak{u}(z^+) = (u_0,\ \dots,\ u_{p-1},\ u)\ , \qquad (35)$$

$$\mathfrak{y}(z^+) = (y_0,\ \dots,\ y_{p-1},\ y_p,\ y)\ . \qquad (36)$$

From equation (32) for all solutions $(\mathbf{u},\ \mathbf{x},\ \mathbf{y})$ of Σ_{sis} and all $k \in \mathbb{N}_0$ it holds:

$$\mathfrak{z}_{k+1}(\mathbf{u},\ \mathbf{y}) \in S(\mathfrak{z}_k(\mathbf{u},\ \mathbf{y}),\ u_k)\ . \qquad (37)$$

We therefore propose

Definition 2. The triple $\Sigma_{cond} = (S,\ U,\ \mathcal{Z}_r)$ is said to be the *condensed model of order* r of the system Σ_{sis}. A sequence $(\mathbf{u},\ \mathbf{z}) = (u_k,\ z_k)_{k\in\mathbb{N}_0}$ is a *solution* of Σ_{cond} if $z_{k+1} \in S(z_k,\ u_k)$ holds for all $k \in \mathbb{N}_0$. □

Lemma 2. For all solutions $(\mathbf{u},\ \mathbf{x},\ \mathbf{y})$ of Σ_{sis} the sequence $(\mathfrak{z}(\mathbf{u},\ \mathbf{y}),\ \mathbf{u})$ is a solution of Σ_{cond}. □

Hence a feedback law solving a control problem stated for the condensed model will go with the original System Σ_{sis} too.

The condensed model is a finite automaton, but it is not necessarily deterministic. When increasing the order r the condensed model is expected to become stronger by including less solutions which cannot be extended to solutions of the underlying switched-integrator-system. It may happen, that the condensed model of a sufficient large order becomes deterministic. In [STI 92] and [STI 93] a discrete-event-system modeling of a (more general) hybrid system is proposed, similar to the condensed model of order 1. [STI 92] and [STI 93] treat the question of how the output events are to be generated in order to receive enough information to solve a certain control problem. In the present paper the generation of output events is taken to be fixed by the systems definition. Therefore we will increase the order of the condensed system to achieve our control goal.

For any fixed order the condensed model on principle can be constructed by a computer program, since it depends only on the computable cones $\mathcal{X}_\cdot(\cdot)$. Note that $|\mathcal{Z}_r|$ grows exponentially when increasing the order r. We therefore are restricted to comparatively small orders r due to limited computer memory and performance.

4. LOCATING CYCLIC TRAJECTORIES

Focus on a given cyclic sequence of input- and output-events $(\mathbf{u}^c,\ \mathbf{y}^c) = (u^c_k,\ y^c_k)_{k\in\mathbb{N}_0} = (u^c_{k+l},\ y^c_{k+l})_{k\in\mathbb{N}_0}$, $u^c_k \in U,\ y^c_k \in \{-1,0,1\}^n$. Further let $(z^c_k)_{k\in\mathbb{N}_0} = \mathbf{z}^c := \mathfrak{z}(\mathbf{u}^c\ ,\mathbf{y}^c)$. Assume the order r of the condensed model to be greater or equal to the cycles length l. We state conditions for the existence of solutions $(\mathbf{u}^c,\ \mathbf{x},\ \mathbf{y}^c)$ of Σ_{sis}. That is, we are asking for trajectories which are cyclic w.r.t. the discrete input/output-behavior. We do not demand the continuous state $\mathbf{x}$ to be cyclic.

To achieve necessary conditions, assume a solution $(\mathbf{u}^c,\ \mathbf{x},\ \mathbf{y}^c)$ to exist. From equation (32) one obtains $x_l \in \mathcal{X}_l(z^c_r)$. Again from (32) it holds $x_l \in \mathcal{X}_0(z^c_{r+l})$. From $r \geq l$ observe $z^c_r = z^c_{r+l}$. This rises $\mathcal{X}_0(z^c_r) \cap \mathcal{X}_l(z^c_r) \neq \emptyset$ as a first necessary condition. Repeatedly applying the above argument yields $x_{kl} \in \mathcal{X}_l(z^c_r),\ x_{kl} \in \mathcal{X}_0(z^c_r)$ for all $k \in \mathbb{N}$. From Lemma 1 obtain

$$x_{kl} = [\Phi^l_{cont}(\cdot,\ (u^c_i)_{i<l})]^k\, x_0\ . \tag{38}$$

Therefore $\mathcal{E} := \{x_{kl}|\ k \in \mathbb{N}\}$ is a subset of $\mathcal{X}_0(z^c_r) \cap \mathcal{X}_l(z^c_r)$ which is invariant w.r.t. $\Phi^l_{cont}(\cdot,\ (u^c_i)_{i<l})$. The existence of such a subset serves as a second necessary condition.

As a sufficient condition for the existence of a solution $(\mathbf{u}^c,\ \mathbf{x},\ \mathbf{y}^c)$ we propose $\emptyset \neq \mathcal{X}_l(z^c_r) \subseteq \mathcal{X}_0(z^c_r)$. For a proof a solution is constructed. Pick any $x_0 \in \mathcal{X}_l(z^c_r)$ and apply the input-sequence $\mathbf{u}^c$: Let for all $k \in \mathbb{N}_0$

$$x_k := \Phi^k_{cont}(x_0,\ (u^c_i)_{i<k})\ , \tag{39}$$

$$y_k := \Phi^k_{disc}(x_0,\ (u^c_i)_{i<k})\ . \tag{40}$$

From $x_0 \in \mathcal{X}_0(z^c_r)$ we know $x_k \neq 0$ and $y_k = y^c_k$ for all $k,\ 0 \leq k \leq r$. From (32) observe $x_l \in \mathcal{X}_l(z^c_r)$. Repeatedly applying the above argument yields $x_{kl} \in \mathcal{X}_0(z^c_r)$ for all $k \in \mathbb{N}$ and therefore $\mathbf{y} = \mathbf{y}^c$. From lemma 1 $(\mathbf{u}^c,\ \mathbf{x},\ \mathbf{y}^c)$ is indeed a solution of Σ_{sis}. The same construction yields a solution, when starting with any x_0 from a subset of $\mathcal{X}_0(z^c_r) \cap \mathcal{X}_l(z^c_r)$ invariant w.r.t. $\Phi^l_{cont}(\cdot,\ (u^c_i)_{i<l})$. The existence of such a subset therefore is necessary *and sufficient*. Summarize the results as

Lemma 3. Let $(\mathbf{u}^c,\ \mathbf{y}^c) = (u^c_k,\ y^c_k)_{k\in\mathbb{N}_0} = (u^c_{k+l},\ y^c_{k+l})_{k\in\mathbb{N}_0}$, $u^c_k \in U,\ y^c_k \in \{-1,0,1\}^n$ be a cyclic sequence of input- and output-events with cycle length $l \leq r$. Further let $(z^c_k)_{k\in\mathbb{N}_0} = \mathbf{z}^c := \mathfrak{z}(\mathbf{u}^c\ ,\mathbf{y}^c)$. For a solution $(\mathbf{u}^c,\ \mathbf{x},\ \mathbf{y}^c)$ of Σ_{sis} to exist, it is

(i) $\mathcal{X}_0(z^c_r) \cap \mathcal{X}_l(z^c_r) \neq \emptyset$ a necessary criterion,

(ii) $\mathcal{X}_0(z^c_r) \supseteq \mathcal{X}_l(z^c_r) \neq \emptyset$ a sufficient criterion, and

(iii) the existence of a subset $\mathcal{E} \neq \emptyset \subseteq \mathcal{X}_0(z^c_r) \cap \mathcal{X}_l(z^c_r)$ with $\Phi^l_{cont}(\mathcal{E},\ (u^c_i)_{i<l}) \subseteq \mathcal{E}$ a necessary and sufficient criterion. □

As a straightforward method to find all possible cycles with length not exceeding r, all $2l$-tuples of l input and l output events for all $l,\ l \leq r$ are checked. Once the condensed model is set up, (i) and (ii) can be evaluated at comparatively high performance. This allready locates most cycles. The technical details when checking (iii) are beyond the scope of the present paper, and therefore only summarized in short. First a base $v_1, \ldots\ v_n$ of $\mathbb{R}^n$ consisting of generalized eigenvectors of $\Phi^l_{cont}(\cdot,\ (u^c_i)_{i<l})$ restricted on the cone $\mathcal{C} := \mathcal{X}_0(z^c_r) \cap \mathcal{X}_l(z^c_r)$ is to be computed, using standard algorithms. If the closure of $\mathcal{C}$ does not contain an eigenvector belonging to a positive real eigenvalue, $\mathcal{C}$ does not contain any invariant subsets. If $\mathcal{C}$ itself contains an eigenvector belonging to a positive real eigenvalue, $\mathcal{C}$ contains an invariant subset. Since $\mathcal{C}$ is not closed, there may be no eigenvectors belonging to positive eigenvalues in $\mathcal{C}$, while there are some on the closure of $\mathcal{C}$. In the latter case, $\mathcal{C}$ under certain circumstances can be tested to contain no invariant subset by a representation of $\mathcal{C}$ w.r.t. the base $v_1, \ldots\ v_n$.

5. SETTING UP A CONTROL LAW

A map $F\colon \mathcal{Z}_r \to U$ is said to be a *control-* or *feedback law* to the condensed system of order r. A solution $(\mathbf{u},\ \mathbf{z})$ of Σ_{cond} is a *closed loop solution*, whenever $u_k = F(z_k)$ holds for all $k \in \mathbb{N}_0$. When applying F to the original system Σ_{sis}, a solution $(\mathbf{u},\ \mathbf{x},\ \mathbf{y})$ is said to be a closed loop solution, if $u_k = F(\mathfrak{z}_k(\mathbf{u},\ \mathbf{y}))$ holds for all $k \in \mathbb{N}_0$.

Pick a cycle $(\mathbf{u}^c,\ \mathbf{y}^c) = (u^c_k,\ y^c_k)_k = (u^c_{k+l},\ y^c_{k+l})_k,\ l \leq r$, let $(z^c_k)_{k\in\mathbb{N}_0} = \mathbf{z}^c := \mathfrak{z}(\mathbf{u}^c\ ,\mathbf{y}^c)$, and choose some subset $\mathcal{E} \neq \emptyset$ of $\mathcal{X}_0(z^c_r) \cap \mathcal{X}_l(z^c_r)$ such that $\Phi^l_{cont}(\mathcal{E},\ (u^c_i)_{i<l}) \subseteq \mathcal{E}$. Further choose some subset $\mathcal{B}$ of $\mathbb{R}^n$ to be the set of allowed initial conditions. As control problem ask for a controller F, such that every closed loop solution $(\mathbf{u},\ \mathbf{x},\ \mathbf{y})$ of Σ_{sis} starting at an allowed initial condition $x_0 \in \mathcal{B}$, reaches the cycle within a finite number of steps $K(x_0) \in \mathbb{N}_0$:

$$x_{K(x_0)+kl} \in \mathcal{E} \quad \forall\, k\ , \tag{41}$$

$$(u_{K(x_0)+k},\ y_{K(x_0)+k})_{k\in\mathbb{N}_0} = (\mathbf{u}^c,\ \mathbf{y}^c)\ . \tag{42}$$

Definitions and criteria of controllability are not made explicit in this paper. A suitable feedback F is simply assumed to exist. For detailed considerations on controllability of hybrid automata see [TIT 94].

We attend to solve the problem by three tasks: First, find a feedback f which is able to force any initial condition x_0 to be transfered to $\mathcal{E}$ within a finite number of steps. Second, make sure that f is able to acknowledge $\mathcal{E}$ being reached. Third, setup F equal to

f until reaching $\mathcal{E}$ is acknowledged, then switch to applying the cyclic input $\mathbf{u}^c$. Since the third task is obvious, we focus on the first and second ones. W.r.t. a fixed order r we propose the following algorithm A1 in terms of the condensed model:

(Step 1) Let $E := \{z |\ z \in \mathcal{Z}_r,\ \mathcal{X}_p(z) \subseteq \mathcal{E}\}$, $W_0 := E,\ i := 1$.

(Step 2) Let $W_i := \emptyset,\ W := \cup_{j=0}^{i-1} W_j$.

(Step 3) For all $z \in \mathcal{Z}_r \setminus W$ check:
If $\exists\, u \in U$ such that $\emptyset \neq S(z,\ u) \subseteq W_{i-1}$ then let $W_i := W_i \cup \{z\},\ f(z) := u$.

(Step 4) If $W_i \neq \emptyset$
then let $i := i + 1$ and proceed with step 2
else finish.

Clearly the algorithm A1 ends in finite time, since all involved sets are finite and the loop is to be executed not exceeding $|\mathcal{Z}_r|$ times. In step 3 all those $z \in \mathcal{Z}_r \setminus W$ are collected in the set W_i, which can be forced to have a successor in W_{i-1} by applying an input u. When in state $z_0 \in W_i$ and applying input $f(z_0)$ for any successor $z_1 \in S(z_0, f(z_0))$ it holds $z_1 \in W_{i-1}$. Repeatedly using this argument yields, that for every solution $(\mathbf{u},\ \mathbf{z})$ of Σ_{cond} satisfying $z_0 \in W_i$ and $u_k = f(z_k)$ for all $k < i$, it holds $z_i \in E$. As acknowledgement for $\mathcal{E}$ being reached by the continuous state $x(t_k)$ at a certain k we use the test $z_k \in E$. To complete the definition of f we let $f(z) := u_{dum}$ for all $z \notin W$ and some dummy input $u_{dum} \in U$. Setting up the overall control law F as described above, the closed loop trajectories for a $K(x_0) \leq |\mathcal{Z}_r|$ satisfy (41) and (42), if $z_0 \in W$. Therefore the control problem is solved, if W contains all $z \in \mathcal{Z}_0$ with $\mathcal{X}_0(z) \cap \mathcal{B} \neq \emptyset$.

If $K(\cdot)$ turns out to be bounded by some (unknown) K_{max} we can construct the control law from the condensed model if r is sufficiently large. So we will be successful, when using algorithm A2:

(Step 1) Let $B := \{z |\ z \in \mathcal{Z}_0,\ \mathcal{X}_0(z) \cap \mathcal{B} \neq \emptyset\}$, $r := 1$.

(Step 2) Run algorithm A1 on the condensed model of order r .

(Step 3) If step 2 results in $W \not\supseteq B$
then let $r := r + 1$ and proceed with step 2
else finish.

Since A2 is only able to generate controls of finite length within finite time, A2 will not stop running if $K(\cdot)$ is not bounded. Note that the assumption, that a suitable feedback law in form of a finite automaton exists, does not imply $K(\cdot)$ to be bounded.

In [TIT 94] an algorithm for linear hybrid automata is presented, to move the continuous state inbetween certain locations, using finite controls. W.r.t. this aspect the results achieved so far are similar to those in [TIT 94].

6. EXTENDED CONDENSED MODELS

Focus on a typical situation where $K(\cdot)$ is unbounded, while a controller in form of a finite automaton exists. Take a cyclic input/output sequence $(\mathbf{u}^c,\ \mathbf{y}^c) = (u_k^c,\ y_k^c)_k = (u_{k+l}^c,\ y_{k+l}^c)_k$, and let $(z_k^c)_{k \in \mathbb{N}_0} = \mathbf{z}^c := \mathfrak{z}(\mathbf{u}^c\ , \mathbf{y}^c)$. Assume that for every $K \in \mathbb{N}_0$ there exists a solution $(\mathbf{u},\ \mathbf{x},\ \mathbf{y})$ of Σ_{sis} such that

$$u_k = u_k^c \quad \forall\, k \leq K\ , \tag{43}$$

$$y_k = y_k^c \quad \forall\, k \leq K\ . \tag{44}$$

Further assume no solution $(\mathbf{u}^c,\ \mathbf{x},\ \mathbf{y}^c)$ of Σ_{sis} to exist. In other words: When $\mathbf{u}^c$ is applied to the system, it will produce a cyclic output equal to $\mathbf{y}^c$ for any finite time, depending on the initial condition x_0. But there is no x_0 yielding an output matching $\mathbf{y}^c$ on the whole time-axis. The condensed model of any order $r,\ r \geq l$, therefore includes z_{k+1}^c as successor of z_k^c when applying u_k^c for all k:

$$z_{k+1}^c \in S(z_k^c,\ u_k^c) \quad \forall\, k \in \mathbb{N}_0\ . \tag{45}$$

From the second assumption there must exist at least one $i,\ r \leq i < r + l$ such that $|S(z_i^c,\ u_i^c)| > 1$. The condensed model includes the possibility, that the cycle will be broken at some time. But it does not reflect that this *must* happen. Let $a \in S(z_i^c,\ u_i^c),\ a \neq z_{i+1}^c$ and assume that a is to be reached in order to solve the control problem. Algorithm A1 may then for any order r fail to find a control forcing the condensed state to reach a, while such a control exists for the original system Σ_{sis}. Therefore algorithm A2 will not finish within finite time. The example presented in the following section illustrates the situation.

To overcome this kind of limitations, the condensed model is extended. Let the order $r \in \mathbb{N}$ be fixed. For every cyclic input/output sequence $(\mathbf{u}^c,\ \mathbf{y}^c)$ of length $l \leq r$ (in the above notation), fulfilling condition (i) of lemma 3 while contradicting to condition (ii), a symbolic input $u_\star^c$ is introduced, with the meaning: "Apply the cyclic input $\mathbf{u}^c$ until the output ceases to match $\mathbf{y}^c$". Let $S_\star(z_r^c,\ u_\star^c)$ denote the set of possible successors of z_r^c when $u_\star^c$ is applied:

$$S_\star(z_r^c,\ u_\star^c) := \bigcup_{i=r}^{r+l-1} \left(S(z_i^c,\ u_i^c) \setminus \{z_{i+1}^c\}\right)\ . \tag{46}$$

To complete the definition, let $S_\star(z,\ u_\star^c) := \emptyset$ for all $z \in \mathcal{Z}_r,\ z \neq z_r^c$, and $S_\star(z,\ u) := S(z,\ u)$ for all $u \in U$. Further let $U_\star$ denote the set of all original inputs $u \in U$ and all newly introduced inputs $u_\star^c$. This leads to

Definition 3. The triple $\Sigma_\star = (S_\star,\ U_\star,\ \mathcal{Z}_r)$ is said to be the *extended condensed model of order* r of the system Σ_{sis}. A sequence $(\mathbf{u},\ \mathbf{z}) = (u_k,\ z_k)_{k\in\mathbb{N}_0}$, $u_k \in U_\star,\ z_k \in \mathcal{Z}_r$, is a *solution* of $\Sigma_\star$ if $z_{k+1} \in S_\star(z_k,\ u_k)$ holds for all $k \in \mathbb{N}_0$. □

With the term *run algorithm A1 on the extended condensed model of order* r we denote running A1 after replacing S by $S_\star$ and U by $U_\star$ in step 3. Define analogous to A2 algorithm A3:

(Step 1) Let $B := \{z|\ z \in \mathcal{Z}_0,\ \mathcal{X}_0(z) \cap \mathcal{B} \neq \emptyset\}$, $r := 1$.

(Step 2) Run algorithm A1 on the extended condensed model of order r .

(Step 3) If step 2 results in $W \not\supseteq B$ then let $r := r + 1$ and proceed with step 2 else finish.

Observe that applying a feedback $F\colon \mathcal{Z}_r \rightarrow U_\star$ to the System Σ_{sis} can be realized by a finite automaton, generating the systems input events $u_k \in U$ by its output events $y_k \in \{-1, 0, 1\}^n$.

7. EXAMPLE

Consider a vehicle moving around within a plane and model its position by the state variable $\xi \in \mathbb{R}^2$. Assume, that it can only move at one of the velocities

$$\mu_1 = (-1,\ -1)^\top\ ,\quad \mu_2 = (1,\ -1)^\top\ , \tag{47}$$
$$\mu_3 = (1,\ 1)^\top\ ,\quad \mu_4 = (-1,\ 1)^\top\ . \tag{48}$$

Further compare the first component of ξ with 0 and the second with 0 and 1 in order to generate output events. Figure 1 shows the state space $\mathbb{R}^2$, the possible velocities of the vehicle, and the comparison value 1 of the second state component.

Introduce the state space transformation

$$T\colon \mathbb{R}^2 \rightarrow \mathbb{R}^3\ , \tag{49}$$
$$(\xi^{(1)},\ \xi^{(2)})^\top \mapsto (\xi^{(1)},\ \xi^{(2)},\ \xi^{(2)} - 1)^\top\ , \tag{50}$$

and choose x to denote the transformed state variable $T(\xi)$. Further let $U := \{v_1,\ v_2,\ v_3,\ v_4\}$ denote the set of corresponding velocities:

$$v_1 = (-1,\ -1,\ -1)^\top,\ v_2 = (1,\ -1,\ -1)^\top, \tag{51}$$
$$v_3 = (1,\ 1,\ 1)^\top,\qquad v_4 = (-1,\ 1,\ 1)^\top. \tag{52}$$

The scenario is now modeled by the switched-integrator-system Σ_{sis} according to definition 1. Since x is known to be in $\mathcal{T} := T(\mathbb{R}^2) \subset \mathbb{R}^3$, we restrict the considerations of the preceding sections to the affine subspace $\mathcal{T}$. Figure 2 shows the continuous time output y when the continuous state x is within the indicated areas in $\mathcal{T}$. Note that beside y_0 all output *events* $y_k = y(t_k)$ hold exactly one component with value zero.

Since $|\mathcal{Z}_1| = 27 + 4 \times 27^2 = 2943$ we cannot list the whole condensed model even for the order 1. We therefore explain some crucial points that occur when the proposed algorithms are executed. The first task is to find a cyclic solution, which then serves as control goal.

Most of the sets $\mathcal{X}_0(z)$ turn out to be empty. Furthermore the restriction on $\mathcal{T}$ clearly marks all output events y with $y^{(3)} > y^{(2)}$ as impossible. Generating the condensed model of *order 1* by a computer program as proposed in section 3 yields only 67 nontrivial states. We give a list of those, which are involved with a cycle we guessed from figure 1. Hereby condensed states $z \in \mathcal{Z}_r$ are written within brackets, input and output events separated by semicolons. Further let $w_1 := (1, 1, 0)^\top,\ w_2 := (1, 0, -1)^\top$ and $w_3 := (0, 1, -1)^\top$ for shortness (see figure 2).

$$S([w_1], v_2) = \{[w_1, w_2;\ v_2]\}\ , \tag{53}$$
$$S([w_2],\ v_4) = \{[w_2, w_1;\ v_4],\ [w_2, w_3;\ v_4]\}\ , \tag{54}$$
$$S([w_1, w_2;\ v_2],\ v_4) = \{[w_2, w_1;\ v_4]\}\ . \tag{55}$$

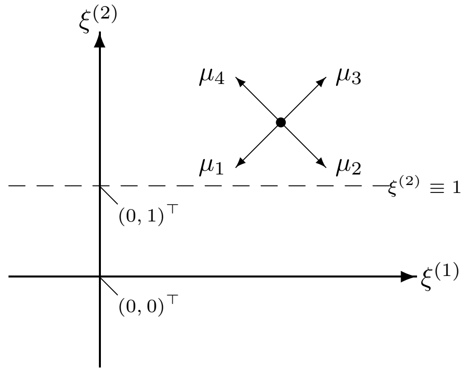

Fig. 1. The vehicles state space

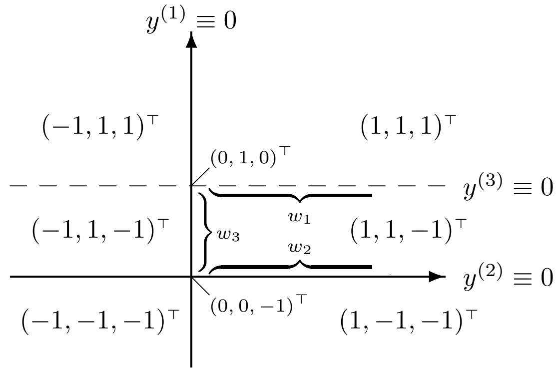

Fig. 2. Output y for $x \in \mathcal{T}$

From (53) and (55) one obtains for the condensed model of *order* 2:

$$S([w_1, w_2, w_1;\ v_2, v_4], v_2) = \{[w_2, w_1, w_2;\ v_4, v_2]\}\ , \quad (56)$$

$$S([w_2, w_1, w_2;\ v_4, v_2], v_4) = \{[w_1, w_2, w_1;\ v_2, v_4]\}\ , \quad (57)$$

hence

$$\mathcal{X}_2([w_2, w_1, w_2;\ v_4, v_2]) \subseteq \mathcal{X}_0([w_2, w_1, w_2;\ v_4, v_2])\ . \quad (58)$$

Let

$$(\mathbf{u}^{cc},\ \mathbf{y}^{cc}) := ((v_4, v_2, v_4, v_2,\ \ldots),\ (w_2, w_1, w_2, w_1,\ \ldots))\ . \quad (59)$$

Then by lemma 3 one obtains cyclic solutions with input $\mathbf{u}^{cc}$ and output $\mathbf{y}^{cc}$ to exist. Further we detect $\Phi^2_{cont}(x,\ (v_4, v_2)) = x$ for all $x \in \mathcal{X}_2([w_2, w_1, w_2;\ v_4, v_2])$. This implies is the continuous state to be cyclic too, whenever the input/output-events match $(\mathbf{u}^{cc},\ \mathbf{y}^{cc})$. Let

$$\mathcal{E} := \{x|\ x \in \mathbb{R}^3,\ -2x^{(3)} > x^{(1)} > -x^{(3)}\} \cap \mathcal{X}_2([w_2, w_1, w_2;\ v_4, v_2]) \quad (60)$$

and check

$$T^{-1}(\mathcal{E}) = \{\xi|\ \xi \in \mathbb{R}^2,\ 2 > \xi^{(1)} > 1,\ \xi^{(2)} = 0\}\ . \quad (61)$$

Let $\mathcal{B} := \mathcal{T}$ and set up a control law as described in section 5: We are looking for a feedback, assuring that every closed loop trajectory reaches $\mathcal{E}$ in finite time and then matches the cycle $(\mathbf{u}^{cc},\ \mathbf{y}^{cc})$.

From the condensed model of order 1 algorithm A1 is able to find a feedback controlling all states $x \in \mathcal{X}_0(w_3)$ to reach $\mathcal{E}$. Now view an arbitrary state $x \in \mathcal{X}_0(w_1)$ and let

$$(\mathbf{u}^{cx},\ \mathbf{y}^{cx}) := ((v_1, v_4, v_1, v_4,\ \ldots),\ (w_1, w_2, w_1, w_2,\ \ldots))\ . \quad (62)$$

When in state x and applying the inputs $\mathbf{u}^{cx}$, the outputs $\mathbf{y}^{cx}$ will occur for some finite time, until w_3 is generated. Note that this finite time depends on x and is not bounded. Algorithm A2 therefore cannot find a feedback achieving all states $x \in \mathcal{X}_0(w_1)$ to reach $\mathcal{X}_0(w_3)$. But the extended condensed model of order 2 as proposed in section 6 includes a symbolic input $u^{cx}_\star$, needed by the above task.
Since all states can be controlled to reach $\mathcal{X}_0(w_1)$ applying at most three inputs, algorithm A3 will find the desired feedback for some $r \leq 3$. In fact A3 already is successful at $r = 2$, using another symbolic input to control $\mathcal{X}_0((-1, 1, 0)^\top)$ to reach $\mathcal{X}_0(w_3)$.

8. CONCLUSIONS

Condensed models have been proposed to describe the external behavior of a given switched-integrator-system by not necessarily deterministic finite automata. Based on condensed models an algorithm has been established, generating a control law in order to force a cyclic trajectory being reached within finite time. The control law can be realized by a finite automaton acting as output feedback on the switched-integrator-system. The number of controls necessary to reach the cycle is not restricted to be uniformly bounded w.r.t. to the continuous initial condition.

REFERENCES

[ALU 93] R. ALUR, C. COURCOUBETI, T.A. HENZINGER, P.-H. HO. “Hybrid automata: an algorithmic approach to the specification and verification of hybrid systems”. In: *R.L. Grossman, A. Nerode, A.P. Rvan, H. Rischel (eds.): Hybrid systems (LNCS 736)*. pp. 209-229, Springer: Berlin.

[BRA 96] M.S. BRANICKY, “On a class of general hybrid dynamical systems”. In: *J. Gertler et al. (eds.): Preprints of the 13th World Congress of IFAC*. San Francisco, Vol. J, 287-292.

[CHA 93] C. CHASE, J. SERRANO, P. RAMADGE. “Periodicity and chaos from switched flow systems: Contrasting examples of discretely controlled continuous systems”. In: *IEEE Transactions on Automatic Control*. Vol. 38, no. 1, pp. 70-83.

[ENG 97] S. ENGELL. “Modelling and analysis of hybrid systems”. In: *Proceedings of the 2nd International Symposium on Mathematical Modelling*. Vienna, 17-31.

[STI 92] J. A. STIVER, P. J. ANTSAKLIS. “Modeling and analysis of hybrid control systems”. In: Proceedings of the 31st Conference on Decision and Control. Tuscon, Arizona. pp. 3748-3751.

[STI 93] J. A. STIVER, P. J. ANTSAKLIS. “State Space partitioning for hybrid control systems”. In: Proceedings of the American Control Conference. San Francisco. pp. 2303-2304.

[TIT 94] M. TITTUS, B. EGARDT. “Control-law synthesis for linear hybrid systems”. In: *Proceedings of the 33rd IEEE Conference on Decision and Control*. Lake Buena Vista, 961-966.

[YU 96] G.-X. YU, P. VAKILI. “Periodic and chaotic dynamics od a switched-server system under corridor policies”. In: *IEEE Transactions on Automatic Control*. Vol. 41, no. 4, pp. 584-588.